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A linear algebraic view of partition regular matrices

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ABSTRACT

Rado showed that a rational matrix is partition regular over $\mathbb{N}$ if and only if it satisfies the columns condition. We investigate linear algebraic properties of the columns condition, especially for oriented (vertex-arc) incidence matrices of directed graphs and for sign pattern matrices. It is established that the oriented incidence matrix of a directed graph Γ has the columns condition if and only if Γ is strongly connected, and in this case an algorithm is presented to find a partition of the columns of the oriented incidence matrix with the maximum number of cells. It is shown that a sign pattern matrix allows the columns condition if and only if each row is either all zeros or the row has both a $+$ and $-$.

1. Introduction

Partition regular matrices are the coefficient matrices associated with those systems of linear homogeneous equations for which, given any finite coloring of $\mathbb{N}$, there is always a monochromatic solution to the system. In his 1933 thesis [10], Richard Rado characterized all finite partition regular matrices as matrices that satisfy the columns condition (defined below). Since then, most of the study of partition regular matrices has taken place in the field of Ramsey theory, with a focus on the

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combinatorial understanding of partition regularity (see [7,8] for surveys of partition regular matrices). In that context, the columns condition serves primarily as a mechanism for checking whether or not a given matrix is partition regular.

This preliminary study suggests that the columns condition is mathematically rich and interesting in its own right, beyond the context in which it originally emerged. Here we apply linear algebraic techniques and combinatorial matrix theory to matrices satisfying the columns condition. In doing so, we investigate the minimum and maximum number of cells possible for a partition used to satisfy the columns condition (Section 2). We consider which matrices associated with a given graph or directed graph satisfy the columns condition (Section 3), as well as which sign patterns allow the columns condition (Section 4). Thus we establish new correspondences between partition regularity and some of the linear algebraic properties of matrices satisfying the columns condition. We hope these results will prove useful to the broader study of partition regularity.

Let $A = [a_{ij}] \in \mathbb{Q}^{v\times u}$ and let $\mathbf{a}_j$ denote the jth column of A. The matrix A has the *columns condition* if there exists a partition $\{I_1, \ldots, I_m\}$ of $\{1, \ldots, u\}$ of A, such that for all $t = 1, \ldots, m$, $I_t \neq \emptyset$ and

$$\sum_{i\in I_t} \mathbf{a}_i \in \operatorname{Span}\left(\left\{\mathbf{a}_j : j \in \bigcup_{k=1}^{t-1} I_k\right\}\right),$$

where the span of the empty set is $\mathbf{0}$; in this case we also say A has the columns condition with $\mathcal{I}$. We refer to A as having $\mathrm{CC}(m)$ if A satisfies the columns condition with a partition consisting of m classes. The *columns condition numbers* of A are the positive integers m such that A has $\mathrm{CC}(m)$.

Rado's theorem [10] states that A has $\mathrm{CC}(m)$ for some $m \in \mathbb{N}$ if and only if for any finite coloring of $\mathbb{N}$, there is a monochromatic solution to $A\mathbf{x} = \mathbf{0}$. That is, A satisfies the columns condition if and only if A is partition regular.

2. Linear algebraic properties of CC(m) matrices

In this section, we investigate linear algebraic properties of matrices having the columns condition, including an examination of the nullspace (kernel) and rank, minimum and maximum columns condition numbers, and for square matrices, matrix powers and spectral properties. The following notation will be used. The *nullspace* of a matrix $A \in \mathbb{Q}^{v\times u}$ is

$$\mathrm{NS}(A) = \{\mathbf{x} \in \mathbb{Q}^u : A\mathbf{x} = 0\},$$

and the *left nullspace* of A is $\mathrm{LNS}(A) = \{\mathbf{x} \in \mathbb{Q}^v : \mathbf{x}^T A = 0\}$. The *nullity* of A, denoted by null A, is the dimension of the nullspace of A.

Observation 2.1. Let $A \in \mathbb{Q}^{v\times u}$. Then A has CC(1) if and only if $A\mathbb{1} = \mathbf{0}$, where $\mathbb{1} = [1, \ldots, 1]^T$.

Observation 2.2. Let $A \in \mathbb{Q}^{v\times u}$ have $\mathrm{CC}(m)$ with some partition $\mathcal{I} = \{I_1 \ldots, I_m\}$ of $\{1, \ldots, u\}$. Then for each row $i = 1, \ldots, v$, either row i consists entirely of zeros, or there exist s, t with $1 \leqslant s, t \leqslant u$ such that $a_{is} > 0$ and $a_{it} < 0$. The same property is true of I_1: for each row $i = 1, \ldots, v$, either $a_{ij} = 0$ for all $j \in I_1$, or there exist $s, t \in I_1$ such that $a_{is} > 0$ and $a_{it} < 0$.

Theorem 2.3. *Let $A \in \mathbb{Q}^{v\times u}$ and let $\mathcal{I} = \{I_1, \ldots, I_m\}$ be a partition of $\{1, \ldots, u\}$. The matrix A has the columns condition with $\mathcal{I}$ if and only if there are vectors $\mathbf{v}^{(t)} = [v_i^{(t)}] \in \mathrm{NS}(A)$, $t = 1, \ldots, m$ with $v_i^{(t)} = 1$ if $i \in I_t$ and $v_i^{(t)} = 0$ if $i \in I_s$ and $s > t$. If A has $\mathrm{CC}(m)$, then* rank $A \leqslant u - m$.

Proof. Suppose $A \in \mathbb{Q}^{v\times u}$ has $\mathrm{CC}(m)$ with some partition $\mathcal{I} = \{I_1, \ldots, I_m\}$ of $\{1, \ldots, u\}$. From the definition of the columns condition, there exist $\alpha_j \in \mathbb{Q}$ such that

$$\sum_{i\in I_t} \mathbf{a}_i = \sum_{j\in\cup_{s=1}^{t-1} I_s} \alpha_j \mathbf{a}_j$$

so define the vector $\mathbf{v}^{(t)} = [v_i^{(t)}] \in \mathrm{NS}(A)$ by

$$v_i^{(t)} = \begin{cases} 1 & \text{if } i \in I_t, \\ -\alpha_i & \text{if } i \in I_s \text{ for } s < t, \\ 0 & \text{if } i \in I_s \text{ for } s > t. \end{cases} \tag{1}$$

Given vectors $\mathbf{v}^{(t)} = [v_i^{(t)}] \in \mathrm{NS}(A), i = 1, \ldots, t$ with $v_i^{(t)} = 1$ if $i \in I_t$, and $v_i^{(t)} = 0$ if $i \in I_s$ with $s > t$, we reverse the process above to establish that with $\mathcal{I}$ the matrix A satisfies the columns condition.

Since $v_i^{(t)} = 0$ for all $i \in \bigcup_{j=t+1}^{m} I_j$ and $v_i^{(t)} = 1$ for $i \in I_t$, the vectors $\mathbf{v}^{(t)}, t = 1, \ldots, m$ are linearly independent, and $\dim \mathrm{NS}(A) \geqslant m$. The statement about the rank is then clear. $\square$

In the case $\mathcal{I}$ is a *consecutive* partition (i.e., $I_1 = \{1, \ldots, k_1\}, I_2 = \{k_1 + 1, \ldots, k_2\}, \ldots, I_m = \{k_{m-1} + 1, \ldots, u\}$), the null vectors $\mathbf{v}^{(t)}$ in Theorem 2.3 take the block form

$$\mathbf{v}^{(1)} = \begin{bmatrix} \mathbb{1} \\ \mathbf{0} \\ \\ \vdots \\ \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{v}^{(2)} = \begin{bmatrix} \mathbf{x}_1^{(2)} \\ \mathbb{1} \\ \mathbf{0} \\ \\ \vdots \\ \\ \mathbf{0} \end{bmatrix}, \ldots, \quad \mathbf{v}^{(t)} = \begin{bmatrix} \mathbf{x}_1^{(t)} \\ \vdots \\ \mathbf{x}_{t-1}^{(t)} \\ \mathbb{1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \ldots, \quad \mathbf{v}^{(m-1)} \begin{bmatrix} \mathbf{x}_1^{(m-1)} \\ \\ \vdots \\ \\ \mathbf{x}_{m-2}^{(m-1)} \\ \mathbb{1} \\ \mathbf{0} \end{bmatrix},$$

$$\mathbf{v}^{(m)} = \begin{bmatrix} \mathbf{x}_1^{(m)} \\ \\ \vdots \\ \\ \mathbf{x}_{m-1}^{(m)} \\ \mathbb{1} \end{bmatrix}.$$

An index k is a *null index* of the matrix $A \in \mathbb{Q}^{v \times u}$ if for every vector $\mathbf{x} = [x_i] \in \mathrm{NS}(A)$, $x_k = 0$. It is well known that a rational matrix A does not have a null index if and only if there is a vector in $\mathrm{NS}(A)$ having every entry nonzero.

Proposition 2.4. *If a matrix $A \in \mathbb{Q}^{v \times u}$ has the columns condition, then A does not have a null index.*

Proof. Assume A has $\mathrm{CC}(m)$ with some partition $\{I_1, \ldots, I_m\}$. For any k such that $1 \leqslant k \leqslant u$, there exists t such that $k \in I_t$. Since for $\mathbf{v}^{(t)}$, defined as in (1), $\mathbf{v}^{(t)} \in \mathrm{NS}(A)$ and $v_k^{(t)} = 1$, k is not a null index. $\square$

Observation 2.5. Let $A \in \mathbb{Q}^{v \times u}$ such that A does not have a null index. Choose $\mathbf{x} = [x_i] \in \mathrm{NS}(A)$ such that $x_i \neq 0$ for all $i = 1, \ldots, u$ and define $D = \mathrm{diag}(x_1, \ldots, x_u)$. Then the matrix AD is $\mathrm{CC}(1)$. If $v = u$, then $D^{-1}AD$ is also $\mathrm{CC}(1)$.

Corollary 2.6. *If the matrix $A \in \mathbb{Q}^{v \times u}$ has the columns condition, then there is a diagonal matrix D such that AD is* $\mathrm{CC}(1)$.

From a linear algebraic point of view it is natural to ask not just whether $A \in \mathbb{Q}^{v \times u}$ has the columns condition, but to determine the columns condition numbers of A, and more generally conditions for a set of positive integers to be the columns condition numbers of a matrix.

Remark 2.7. If $S = \{\ell, \ell+1, \ldots, m-1, m\}$ is a consecutive set of positive integers, then there exists an integer matrix A that does not have a zero column for which S is the set of columns condition numbers of A. Specifically, for $\ell = m = 1$, let A be the 1×2 matrix $\begin{bmatrix} -1 & 1 \end{bmatrix}$. For $\ell = 1, m = 2$, let A be the 2×4 matrix $\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. For $\ell = 1$ and $m > 2$, let A be the $1 \times (m+1)$ matrix $\begin{bmatrix} -1, 1, 1, \ldots, 1, -m+2 \end{bmatrix}$. For $1 < \ell$ let $A = [a_{ij}]$ be the $\ell - 1 \times (m + \ell - 1)$ matrix defined as follows:

- For $j = 1, \ldots, \ell - 1$,
  - $a_{j,2(j-1)+1} = -1$.
  - For $i < j$, $a_{i,2(j-1)+1} = 1$.
  - For $i > j$, $a_{i,2(j-1)+1} = 0$.
  - For $i \leqslant j$, $a_{i,2(j-1)+2} = 1$.
  - For $i > j$, $a_{i,2(j-1)+2} = 0$.
- For $j = 2\ell - 1, \ldots, m + \ell - 1$, for $i = 1, \ldots, \ell$, $a_{ij} = 1$.

For example, with $S = \{3, 4, 5, 6, 7\}$,

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Question 2.8. If A has $\mathrm{CC}(\ell)$ and $\mathrm{CC}(m)$ with $\ell < m$, does A necessarily have $\mathrm{CC}(k)$ for $\ell < k < m$?

The following result provides a partial answer to this question.

Theorem 2.9. *If $A \in \mathbb{Q}^{v \times u}$ has $\mathrm{CC}(1)$ and $\mathrm{CC}(m)$ with $1 < m$, then A has $\mathrm{CC}(k)$ for $1 \leqslant k \leqslant m$.*

Proof. Let $\mathbf{a}_j$ denote the jth column of A and assume A has $\mathrm{CC}(1)$ and $\mathrm{CC}(m)$. Since A has $\mathrm{CC}(m)$, A satisfies the columns condition with a partition $\{I_1, \ldots, I_m\}$ of $\{1, \ldots, u\}$. For each k, define a new partition $\{I'_1, \ldots, I'_k\}$ by $I'_j = I_j$ for $j = 1, \ldots, k-1$ and $I'_k = \cup_{t=k}^m I_t$. Since A has $\mathrm{CC}(m)$, for all $t = 1, \ldots, k-1$

$$\sum_{i \in I_t} \mathbf{a}_i \in \operatorname{Span}\left(\left\{\mathbf{a}_j : j \in \bigcup_{s=1}^{t-1} I_s\right\}\right).$$

Since A has $\mathrm{CC}(1)$, $\sum_{j=1}^{u} \mathbf{a}_i = \mathbf{0}$. Thus,

$$\sum_{i \in I'_k} \mathbf{a}_i = \sum_{i \in \bigcup_{s=k}^m I_s} \mathbf{a}_i = -\sum_{i \in \bigcup_{s=1}^{k-1} I_s} \mathbf{a}_i \in \operatorname{Span}\left(\left\{\mathbf{a}_j : j \in \bigcup_{s=1}^{k-1} I_s\right\}\right). \quad \square$$

Partition regular matrices need not be square, but for a square partition regular matrix we can study powers of the matrix and the spectrum (multiset of eigenvalues) of the matrix. The next result follows immediately from Theorem 2.3.

Corollary 2.10. *If A is a square matrix that has the columns condition with partition $\mathcal{I} = \{I_1 \ldots, I_m\}$, then A^k has the columns condition with partition $\mathcal{I}$.*

Proposition 2.11. *A multiset Λ of v complex numbers is the spectrum of a complex $v \times v$ matrix that has the columns condition if and only if $0 \in \Lambda$.*

Proof. If $A \in \mathbb{C}^{v \times v}$ has the columns condition, then A has $\mathrm{CC}(m)$ for some m. So by Theorem 2.3, $\operatorname{rank} A \leqslant v - m < v$, so 0 is an eigenvalue of A.

Let Λ be a multiset of v complex numbers such that $0 \in \Lambda$. Denote the elements of Λ by $\lambda_1 = 0, \lambda_2, \ldots, \lambda_v$, and let D be the diagonal matrix having diagonal entries $\lambda_1, \ldots, \lambda_v$. Extend the linearly independent set $\{\mathbb{1}\}$ to a basis $\{\mathbf{s}_1 = \mathbb{1}, \mathbf{s}_2, \ldots, \mathbf{s}_v\}$ for $\mathbb{C}^v$, and let $S = [\mathbf{s}_1 \quad \mathbf{s}_2 \quad \ldots \quad \mathbf{s}_v] \in \mathbb{C}^{v\times v}$. Then $A = SDS^{-1}$ has CC(1). $\square$

A similar argument can be used to construct a matrix that has CC(1) and has any given Jordan canonical form that includes zero as an eigenvalue.

3. CC(m) matrices associated with graphs

A (simple, undirected, finite) *graph* $G = (V, E)$ has a nonempty finite set V of vertices and a set E of edges, where an edge is a two-element subset of vertices. We examine matrices associated with a graph G in the context of the columns condition. If $\{i, j\}$ is an edge of G, we write $i \sim j$, and $\deg i$ denotes the *degree* of vertex i, i.e., the number of edges incident with i. The following matrices are naturally associated with a graph G [5]:

- The *adjacency matrix* $A_G = [a_{ij}]$, where $a_{ij} = 1$ if $i \sim j$ and $a_{ij} = 0$ otherwise.
- The *Laplacian matrix* $L_G = [\ell_{ij}]$, where $\ell_{ii} = \deg i$ and for $i \neq j$, $\ell_{ij} = -1$ if $i \sim j$ and $\ell_{ij} = 0$ otherwise.
- The *signless Laplacian matrix* $L_G = [\ell_{ij}]$, where $\ell_{ii} = \deg i$ and for $i \neq j$, $\ell_{ij} = 1$ if $i \sim j$ and $\ell_{ij} = 0$ otherwise.
- The *Seidel matrix* $S_G = J - I - 2A_G$, where J is the matrix having all entries equal to 1 and I is the identity matrix.
- The *(vertex-edge) incidence matrix* $N_G = [n_{ie}]$, where $n_{ie} = 1$ if vertex i is an endpoint of edge e, and 0 otherwise.
- The oriented (vertex-edge) incidence matrix, defined below.

Adjacency matrices, signless Laplacian matrices, and incidence matrices do not satisfy the columns condition since they are nonnegative and nonzero matrices (see Observation 2.2).

We need some additional graph theoretic definitions. Let $G = (V, E)$ be a graph. The *order* of G, denoted $|G|$, is the number of vertices of G, and the *size* of G is the number of edges of G. A graph $G' = (V', E')$ is a *subgraph* of graph G if $V' \subseteq V, E' \subseteq E$. The subgraph $G[R]$ of G *induced* by $R \subseteq V$ is the subgraph with vertex set R and edge set $\{\{i, j\} \in E | i, j \in R\}$. G is r-regular if for every vertex v of G, $\deg v = r$. A *walk* in G is an alternating sequence $(v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell)$ of vertices and edges (not necessarily distinct), such that v_{i-1} and v_i are endpoints of e_i for $i = 1, \ldots, \ell$. G is *connected* if there exists a walk between any two distinct vertices of G; otherwise it is *disconnected* (a graph of order one is connected). A *(connected) component* of a graph is a maximal connected subgraph. A *cycle* is a walk in which the initial vertex is equal to the final vertex and the vertices are otherwise distinct; a *Hamilton cycle* is a cycle that contains all the vertices of Γ. An edge of a connected graph G is a *bridge* if $G - e$ is disconnected (where $G - e$ denotes the graph obtained from G by deleting edge e).

Observation 3.1. For any graph G, the Laplacian matrix L_G of G has CC(1) since $L_G\mathbb{1} = 0$. For any connected graph G, there is no proper subset of the columns of L_G that sums to zero, so L_G does not have CC(m) for $m > 1$.

Theorem 3.2. *Let G be a graph. The Seidel matrix S_G has* CC(1) *if and only if* $|G| \equiv 1 \bmod 4$ *and G is* $\frac{|G|-1}{2}$*-regular. For any $m > 1$, S_G does not have* CC(m).

Proof. Every row of S_G contains one entry 0, and each of the remaining entries is 1 or -1. If I_1 is a subset of vertices so that the columns with indices in I_1 sum to zero, then every row must have the same number of 1's and -1's in the selected columns. If I_1 is not the entire set of columns, this is impossible, since then some rows will have a zero and others will not, so one or the other type of row must have an odd number of nonzero entries. So S_G does not have CC(m) for $m > 1$. For CC(1), the sum of all columns must be zero, so each row must contain $\frac{|G|-1}{2}$ entries equal to 1 and $\frac{|G|-1}{2}$ entries equal to -1. Thus $|G|$ is odd and G is $\frac{|G|-1}{2}$-regular. If r is odd, it is not possible for a graph of odd order to be r-regular, so $\frac{|G|-1}{2}$ is even and thus $|G| \equiv 1 \mod 4$. $\square$

For a graph G, an *orientation* $\vec{G}$ of G is obtained by assigning a direction to each edge, or equivalently, by replacing each edge $\{i,j\}$ by exactly one of the arcs (i,j), (j,i). The *oriented (vertex-edge) incidence matrix* of $\vec{G}$, hereafter called an *oriented incidence matrix*, is denoted $D_{\vec{G}} = [d_{ie}]$. If $e = (i,j)$, then $d_{ie} = -1$, $d_{je} = 1$, and $d_{ke} = 0$ for $k \neq i, k \neq j$. Some oriented incidence matrices satisfy the columns condition and others do not. In the remainder of this section we characterize oriented graphs $\vec{G}$ such that $D_{\vec{G}}$ satisfies the columns condition and investigate related questions.

Many of the results for oriented graphs are in fact true for all (simple) directed graphs, so we state them for directed graphs. A (simple, finite) *directed graph* $\Gamma = (V, E)$ has a nonempty finite set V of vertices and a set E of arcs, where an arc is ordered pair of distinct vertices (loops are not permitted). Note that an orientation $\vec{G}$ of a graph G is a directed graph that contains at most one of each possible pair of arcs (i,j), (j,i) between vertices i and j. The *oriented incidence matrix* of a directed graph Γ, denoted by $D_\Gamma = [d_{ie}]$ has $d_{ie} = -1$, $d_{je} = 1$, and $d_{ke} = 0$ for $k \neq i, k \neq j$ where $e = (i,j)$. Note that the oriented incidence matrix $D_{\vec{G}}$ of an oriented graph $\vec{G}$ is the same as the oriented incidence matrix of $\vec{G}$ viewed as a directed graph, so the use of the same notation should not cause confusion.

First we need some definitions for directed graphs. The definitions of the following terms are extended in the obvious way from graphs to directed graphs: order, size, sub-directed-graph, induced sub-directed-graph. Let $\Gamma = (V, E)$ be a directed graph. The *in-degree*, denoted in i, (respectively, *out-degree*, denoted out i,) is the number of arcs $(j,i), j \in V$ (respectively, $(i,j), j \in V$). A *walk* in Γ is an alternating sequence $(v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell)$ of vertices and arcs (not necessarily distinct), such that $e_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, \ell$. Γ is *strongly connected* there exists a walk between any two distinct vertices of Γ. (A directed graph of order one is strongly connected by definition.) A *strong component* of a graph is a maximal strongly connected sub-directed-graph. Γ is *connected* if the undirected graph obtained from Γ by ignoring orientation (i.e., replacing arc(s) (v,u) or $(v,u), (u,v)$ by edge $\{v,u\}$) is connected. A *cycle* in Γ is a walk in which the initial vertex is equal to the final vertex and the vertices are otherwise distinct; a *Hamilton cycle* is a cycle that contains all the vertices of Γ. A *path* in G is a walk $(v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell)$ with $v_i \neq v_j$ for $i \neq j$. If Γ has a walk from u to v, then Γ has a path from u to v (by omitting redundancies). A set of vertices S of Γ is a *source* if Γ does not contain any arcs of the form (j,i) with $i \in S$ and $j \in V \backslash S$.

Observation 3.3. For any directed graph Γ, the sum of the entries in each column of the matrix D_Γ is 0, i.e., $\mathbb{1}^T D_\Gamma = 0$. Thus the sum of all the entries in D_Γ is 0. Note that the the sum of the entries in a row is variable.

Theorem 3.4. *Let Γ be a connected directed graph. The oriented incidence matrix of Γ, D_Γ, satisfies the columns condition if and only if Γ is strongly connected.*

Proof. Let Γ be a strongly connected directed graph. Γ is the (nondisjoint) union of its (oriented) cycles, because for every arc (u,w) of Γ, there is a path from w to u, and this path together with (u,w) is a cycle that includes (u,w). Thus the following algorithm produces a partition $\mathcal{I}$ of the column indices of D_Γ that satisfies the columns condition:

1. Choose a cycle C. The first cell $I_1 \in \mathcal{I}$, consists of the arcs in C. Set $k = 1$.
2. Choose an arc $e \notin \bigcup_{j=1}^k I_j$; e is an arc of some cycle C. Define I_{k+1} to be the set of arcs of C not in $\bigcup_{j=1}^k I_j$. Add 1 to k.
3. Repeat step 2 until every arc is in some cell I_k.

In any cycle C of Γ, each vertex v has exactly one arc of the form (u,v) and exactly one arc of the form (v,w). Thus each row of the submatrix of columns of the arcs of C has one 1 and one -1, and the sum of the columns is 0. So the sum of the columns in I_1 is zero. In step 2, C is a cycle, so again the sum of the columns is zero, and the sum of the columns of I_{k+1} is the negative of the sum of the columns of arcs of C in $\bigcup_{j=1}^{k} I_j$. Thus D_Γ satisfies the columns condition with $\mathcal{I}$.

For the converse, we assume $\Gamma = (V, E)$ is not strongly connected and show that for every partition $\{I_1, \dots, I_m\}$, D_Γ does not satisfy the columns condition. Let $\mathbf{d}_e$ denote the column of D_Γ associated with arc e. Since Γ is not strongly connected, Γ has a strongly connected component, $\Gamma[S]$, such that S is a source [11, Fact 29.5.5]. Since Γ is connected and there is no arc from $V\backslash S$ to S, there is at least one arc from S to $V\backslash S$. Let t be the least index such that an arc from S to $V\backslash S$ is in I_t. So for $r < t$, every arc e in I_r has both ends in S or both ends in $V\backslash S$, and thus $\sum_{i\in S}(\mathbf{d}_e)_i = 0$. So for any $\alpha_e \in \mathbb{Q}$,

$$\sum_{i\in S}\left(\sum_{e\in\cup_{r<t}I_r}\alpha_e\mathbf{d}_e\right)_i = 0.$$

But because I_t has one or more arcs from S to $V\backslash S$ and none from $V\backslash S$ to S,

$$\sum_{i\in S}\left(\sum_{e\in I_t}\mathbf{d}_e\right)_i < 0,$$

so $\sum_{e\in I_t}\mathbf{d}_e$ is not a linear combination of $\{\mathbf{d}_f : f \in \cup_{r<t}I_r\}$ and $D_{\vec{G}}$ does not satisfy the columns condition with any partition. $\square$

Observation 3.5. Let Γ be a strongly connected directed graph. If $\mathcal{I} = \{I_1, \dots, I_k\}$ is a partition of the column indices of D_Γ that satisfies the columns condition, then I_1 is an arc-disjoint union of cycles.

We explore the minimum and maximum values m for which the oriented incidence matrix D_Γ of a directed graph Γ has CC(m). To do so, we need some additional terminology and known results. If $A \in \mathbb{Q}^{v\times u}$, $R \subseteq \{1, 2, \dots, v\}$ and $C \subseteq \{1, 2, \dots, u\}$, then $A[R|C]$ denotes the *submatrix* of A whose rows and columns are indexed by R and C, respectively. Let $\Gamma = (V, E)$ be a directed graph. If $W \subset V$ and $v \notin W$, an *external* path from v to W (respectively, from W to v) is a path $(v_0 = v, e_1, v_1, \dots, v_k, e_{k+1}, v_{k+1} = w)$ (respectively, $(v_0 = w, e_1, v_1, \dots, v_k, e_{k+1}, v_{k+1} = v)$) such that $w \in W$ and for $i = 1, \dots, k, v_i \notin W$ (note that it is possible $k = 0$). If Γ is strongly connected, then for any nonempty set of vertices W and $v \notin W$, there are external paths $(w, v_1, \dots, v_j, v)$ from W to v and $(v, u_1, \dots, u_i, w')$ from v to W. from V_k to v and from v to V_k. If $\{v_1, \dots, v_j\} \cap \{u_1, \dots, u_i\} \neq \emptyset$, let t be the first index such that $v_t \in \{u_1, \dots, u_i\}$ and replace v by v_t. Thus when choosing such external paths in Algorithm 3.8 below, we may assume that $\{v_1, \dots, v_j\} \cap \{u_1, \dots, u_i\} = \emptyset$.

As noted in Observation 3.3, $\mathbb{1}$ is a left null vector of D_Γ, so from the next (well known) result we see that for a connected digraph Γ, $\{\mathbb{1}\}$ is a basis for the left nullspace of D_Γ.

Theorem 3.6 [4, Theorem 8.3.1]**.** *If G is a connected graph, then for any orientation $\vec{G}$ of G,*

$$\text{rank } D_{\vec{G}} = |G| - 1.$$

Corollary 3.7. *Let Γ be a connected directed graph of order v and size u. Then* rank $D_\Gamma = v - 1$ *and* null $D_\Gamma = u - v + 1$.

The following algorithm produces a partition with the maximum number of cells to have D_Γ satisfy the columns condition.

Algorithm 3.8. Let Γ be a strongly connected directed graph of order v and size u.

1. Choose a cycle $C_1 = (V_1, E_1)$.

   (a) The first cell I_1 of the partition is the set E_1 of arcs of C_1.
   (b) Define $D_1 = D_\Gamma[V_1|E_1]$.
   (c) Set $k = 1$.

2. If $V \neq V_k$, choose a vertex $v \notin V_k$ and external paths $(w, v_1, \ldots, v_j, v)$ from V_k to v and $(v, u_1, \ldots, u_i, w')$ from v to V_k with $\{v_1, \ldots, v_j\} \cap \{u_1, \ldots, u_i\} = \emptyset$.
   (a) Set $V_{k+1} = V_k \cup \{v_1, \ldots, v_j, v, u_1, \ldots, u_i\}$,
   (b) Set $I_{k+1} = \{(w, v_1), (v_1, v_2), \ldots, (v_{j-1}, v_j), (v_j, v), (v, u_1), (u_1, u_2) \ldots, (u_{i-1}, u_i), (u_i, w')\}$.
   (c) Set $E_{k+1} = E_k \cup I_{k+1}$.
   (d) Define $D_{k+1} = D_\Gamma[V_{k+1}|E_{k+1}]$.
   (e) Add 1 to k.
3. Repeat step 2 until all vertices are in V_k. Set $\ell = k$.
4. If $E \neq E_k$, choose one arc $e \notin E_k$.
   (a) Set $I_{k+1} = \{e\}$.
   (b) Set $E_{k+1} = E_k \cup I_{k+1}$.
   (c) Define $D_{k+1} = D_\Gamma[V|E_{k+1}]$.
   (d) Add 1 to k.
5. Repeat step 4 until arcs are in some cell I_k. Set $m = k$.

Theorem 3.9. *Let Γ be a strongly connected directed graph of order v and size u. Algorithm 3.8 produces a partition $\mathcal{I}$ of $\{1, \ldots, u\}$ into $m = u - v + 1$ cells so that D_Γ satisfies the columns condition with $\mathcal{I}$.*

Proof. We show that null $D_k = k$ for $k = 1, \ldots, m = u - v + 1$: consider first the stages $1 \leqslant k \leqslant \ell$ (where vertices are added). Since a cycle has the same number of arcs as vertices and at each stage after the first the number of arcs added is one more than the number of vertices, the number of columns of $D_k = D_\Gamma[V_k|E_k]$ is $|V_k| + k - 1$. By Corollary 3.7, rank $D_k = |V_k| - 1$, so null $D_k = k$. For the remaining stages, one arc is added at each stage, so the nullity increases by one, i.e., null $D_k = k$ for $k = 1, \ldots, m$. Since $\ell = |E_\ell| - v + 1$ and there are $u - |E_\ell|$ edges to add after stage ℓ, $m = u - v + 1$.

At each stage $1 \leqslant k \leqslant m$, the induced directed graph $\Gamma[V_k]$ is strongly connected, so there is a path from w' to w, where w and w' are the ends of the external paths in step 2, or $e = (w, w')$ in step 4. Let C_k be the cycle that is the union of the paths from w' to w, from w to v, and from v to w' (step 2), or the path from w' to w together with $e = (w, w')$ (step 4). The sum of the columns associated with arcs in the cycle C_k is 0, so D_k satisfies the columns condition with the partition $\mathcal{I}_k$. Thus D_Γ satisfies the columns condition with the partition $\mathcal{I}_m$. $\square$

Example 3.10. Let Γ be the directed graph (or oriented graph) shown in Fig. 1. With the arcs in alphabetical order, the oriented incidence matrix is

$$D_\Gamma = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

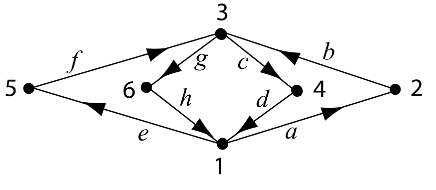

Fig. 1. The directed graph Γ for Example 3.10.

We apply Algorithm 3.8 to D_Γ:

1. Choose the cycle (1,2,3,4), so $V_1 = \{1, 2, 3, 4\}$ and $I_1 = E_1 = \{a, b, c, d\}$.
2. Choose $v = 5$, so $V_2 = \{1, 2, 3, 4, 5\}$ and $I_2 = \{e, f\}$ and $E_2 = \{a, b, c, d, e, f\}$.
3. Choose $v = 6$, so $V_3 = \{1, 2, 3, 4, 5, 6\}$ and $I_3 = \{g, h\}$ and $E_3 = \{a, b, c, d, e, f, g, h\}$.

Thus D_Γ has CC(3) (note $3 = 6 - 4 + 1$).

Remark 3.11. If G has a bridge, then obviously G cannot be oriented to be strongly connected. Let G be a connected graph that does not have a bridge. It is not difficult to see that an orientation can be chosen for G that makes G strongly connected: since G does not have a bridge, every edge of G lies on a cycle (see, for example [3, p. 19]). Then the method in Algorithm 3.8 can be used to orient G to be strongly connected, by keeping the oriented subgraph always strongly connected. In step 1, select an unoriented cycle and orient it to be an oriented cycle. In step 2 select (unoriented) external paths to/from the vertex v to be added to the oriented part (by using a cycle that contains v and at least one vertex from the oriented part – v can be chosen so its cycle contains a vertex from the oriented part) and orient the external paths from w to v to w' to be one oriented path. In step 4, either orientation may be chosen for the newly oriented arc e. If G is not connected, G can be oriented so that $D_{\vec{G}}$ satisfies the columns condition if and only if every connected component of G has no bridges.

Question 3.12. Let D_Γ be the oriented incidence matrix of a directed graph Γ. What is $\min\{m : D_\Gamma \text{ has } \mathrm{CC}(m)\}$?

Remark 3.13. If $\ell < u - v + 1$, then the proof of Theorem 3.9 can be modified to show that D_Γ has CC(k) for all $k = \ell + 1, \dots, u - v + 1$ (where ℓ is the index at which all the vertices have been added in Algorithm 3.8). However, the matrix D_Γ in Example 3.10 also has CC(1) and CC(2), and we do not see a natural way to adapt the algorithm to find this.

A partial answer to Question 3.12 is provided by the following results.

Observation 3.14. Let Γ be a directed graph of order v and size u. Then D_Γ has CC(1) if and only if for every vertex v of Γ, $\mathrm{in}(v) = \mathrm{out}(v)$.

It is well known (see, for example [1, Theorem 12.1.2]) that for every vertex v of a connected directed graph Γ, $\mathrm{in}(v) = \mathrm{out}(v)$ if and only if Γ has a closed Euler trail (a *closed Euler trail* is a walk that ends at the same vertex at which it began and includes every arc exactly once).

Theorem 3.15. *Let Γ be a directed graph of order v and size u that contains a Hamiltonian cycle and at least one additional arc. Then D_Γ has* CC(k) *for $k = 2, \dots, u - v + 1$. Furthermore, if D_Γ has* CC(k), *then $k \leqslant u - v + 1$.*

Proof. Let C be a Hamilton cycle of Γ. Let k be such that $2 \leqslant k \leqslant u - v + 1$. Define I_1 to be the indices of arcs in C. For $t = 2, \dots, k - 1$, define I_t to be a single arc $e \notin \bigcup_{j=1}^{t-1} I_j$. Define I_k to be all the remaining arcs. Thus D_Γ has CC(k).

If D_Γ has CC(k), then by Theorem 2.3, rank $D_\Gamma \leqslant u - k$. Since Γ is connected, by Corollary 3.7, rank $D_\Gamma = v - 1$, so $k \leqslant u - v + 1$. □

As a consequence of Observation 3.14 and Theorem 3.15, for a directed graph Γ of order v and size u that contains a Hamilton cycle, the columns condition numbers of D_Γ are known exactly: if Γ has no arcs other than the Hamilton cycle, D_Γ has CC(1) only. Otherwise, D_Γ has CC(k) for $k = 2, \dots, u - v + 1$, and D_Γ has CC(1) if and only if for every vertex v of Γ, $\mathrm{in}(v) = \mathrm{out}(v)$.

4. Sign pattern matrices which allow CC(m)

A *sign pattern matrix* (or *sign pattern* for short) is a matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\operatorname{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in A. If $\mathcal{Y}$ is an $v \times u$ sign pattern, the *sign pattern class* (or *qualitative class*) of $\mathcal{Y}$, denoted $\mathcal{Q}(\mathcal{Y})$, is the set of all $A \in \mathbb{R}^{v\times u}$ such that $\operatorname{sgn}(A) = \mathcal{Y}$. It is traditional in the study of sign patterns to say that a sign pattern $\mathcal{Y}$ *requires* property P if every matrix in $\mathcal{Q}(\mathcal{Y})$ has property P and to say that $\mathcal{Y}$ *allows* property P if there exists a matrix in $\mathcal{Q}(\mathcal{Y})$ that has property P. See [6] for a survey about sign patterns and [2] for a recent survey of allows properties. Patterns that require the columns condition are too trivial to be of interest, as the next proposition shows.

Proposition 4.1. *The only sign patterns that require the columns condition are the all zero sign patterns.*

Proof. Assume the $v \times u$ sign pattern $\mathcal{Y} = [\psi_{ij}]$ has a nonzero entry. Construct a matrix $A = [a_{ij}] \in \mathcal{Q}(\mathcal{Y})$ as follows:

- For all i, j such that $\psi_{ij} = 0$, $a_{ij} = 0$.
- For all i, j such that $\psi_{ij} = +$, $a_{ij} = 1$.
- For all i, j such that $\psi_{ij} = -$, $a_{ij} = -\frac{1}{u}$.

There is no subset of columns that sum to zero, so A does not satisfy the columns condition. $\square$

The next observation is a sign pattern version of Observation 2.2.

Observation 4.2. Let $\mathcal{Y} = [\psi_{ij}]$ be a $v \times u$ sign pattern that allows the columns condition with partition $\mathcal{I} = \{I_1, \ldots, I_m\}$. Then for each row $i = 1, \ldots, v$, either row i consists entirely of zeros, or there exist s, t with $1 \leqslant s, t \leqslant u$ such that $\psi_{is} = +$ and $\psi_{it} = -$. The same property is true for I_1: for each row $i = 1, \ldots, v$, either $\psi_{ij} = 0$ for all $j \in I_1$, or there exist $s, t \in I_1$ such that $\psi_{is} = +$ and $\psi_{it} = -$.

The condition that any nonzero row must have at least one $+$ entry and at least one $-$ entry is also sufficient for a sign pattern to allow the columns condition.

Theorem 4.3. *Let $\mathcal{Y}$ be an $v \times u$ sign pattern. The following are equivalent*:

1. *For each row of $\mathcal{Y}$, either the row has at least one $+$ entry and at least one $-$ entry, or every entry of the row is* 0.
2. *$\mathcal{Y}$ allows* CC(1).
3. *$\mathcal{Y}$ allows the columns condition*.

Proof. It is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Assume that for each row of $\mathcal{Y} = [\psi_{ij}]$, either the row has at least one $+$ entry and at least one $-$ entry, or every entry of the row is 0. If row i is not entirely zero, let $n(i)$ denote the least j such that $\psi_{ij} = -$; otherwise, $n(i) = 0$. Construct a matrix $A = [a_{ij}]$ as follows:

- For all i, j such that $\psi_{ij} = 0$, let $a_{ij} = 0$.
- For all i such that $n(i) > 0$:
  - If $\psi_{ij} = +$, then $a_{ij} = 1$.
  - If $\psi_{ij} = -$ and $j > n(i)$, then $a_{ij} = -\frac{1}{u}$.
  - $a_{i,n(i)} = -\sum_{j\neq n(i)} a_{ij}$.

Clearly $A \in \mathcal{Q}(\mathcal{Y})$ and $A\mathbb{1} = \mathbf{0}$, so A has CC(1). $\square$

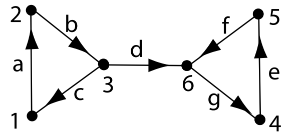

Fig. 2. The oriented graph $\vec{G}$ for Example 4.5.

The *minimum rank* of a $v \times u$ sign pattern $\mathcal{Y}$ is

$$\mathrm{mr}(\mathcal{Y}) = \min\{\operatorname{rank} A : A \in \mathcal{Q}(\mathcal{Y})\},$$

and the *maximum nullity* of $\mathcal{Y}$ is

$$\mathrm{M}(\mathcal{Y}) = \max\{\operatorname{null} A : A \in \mathcal{Q}(\mathcal{Y})\}.$$

Clearly $\mathrm{mr}(\mathcal{Y}) + \mathrm{M}(\mathcal{Y}) = u$. Minimum rank of a sign pattern is called *sign rank* in communication complexity theory (see, for example [9]).

It is not always the case that the nullity of a partition regular matrix can be realized as the number of cells in a partition that achieves the columns condition. For example, for $A = \begin{bmatrix} 3 & -1 & -1 & -1 \end{bmatrix}$, $\operatorname{null} A = 3$ but A has $\mathrm{CC}(m)$ only for $m = 1$. Furthermore, if $\mathcal{Y}$ allows partition regularity, it is not necessary that the maximum nullity be realizable as a columns condition number, as the following example shows.

Example 4.4. Let

$$B = \begin{bmatrix} 1 & -4 & -1 & -2 & -9 \\ -6 & 1 & -1 & -4 & -1 \\ -1 & -1 & 1 & -3 & -6 \\ -1 & -4 & -6 & 1 & -1 \\ -4 & -1 & -4 & -1 & 1 \end{bmatrix}$$

and $\mathcal{Y} = \operatorname{sgn}(B)$. A simple computation shows that $\operatorname{rank} B = 3$. Since every row is nonzero and the unique $+$ in row i is in column i, by Observation 4.2, for $A \in \mathcal{Q}(\mathcal{Y})$, A has $\mathrm{CC}(m)$ only if $m = 1$. Thus

$$\mathrm{M}(\mathcal{Y}) \geqslant \operatorname{null} B = 2 > 1 = \max\{m : A \text{ has } \mathrm{CC}(m) \text{ and } \operatorname{sgn}(A) = \mathcal{Y}\}.$$

Note that $B\mathbb{1} \neq \mathbf{0}$, so B does not satisfy the columns condition.

Recall that if an oriented graph $\vec{G}$ is not strongly connected, then its oriented incidence matrix $D_{\vec{G}}$ does not satisfy the columns condition. However, it is possible that the sign pattern $\operatorname{sgn}(D_{\vec{G}})$ allows the columns condition.

Example 4.5. Let $\vec{G}$ be the oriented graph shown in Fig. 2. Observe that $\vec{G}$ is not strongly connected. With the edges in alphabetical order, the sign pattern of the oriented incidence matrix is

$$\operatorname{sgn}(D_{\vec{G}}) = \begin{bmatrix} - & 0 & + & 0 & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 \\ 0 & + & - & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & + & - & 0 \\ 0 & 0 & 0 & + & 0 & + & - \end{bmatrix}.$$

Since $\operatorname{sgn}(D_{\vec{G}})$ has at least one $+$ and at least one $-$ in every row, by Theorem 4.3, $\operatorname{sgn}(D_{\vec{G}})$ allows the columns condition.

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References

[1] R.A. Brualdi, Introductory Combinatorics, fourth ed., Pearson Prentice-Hall, Upper Saddle River, NJ, 2004.
[2] M. Catral, D.D. Olesky, P. van den Drissche, Allow problems concerning spectral properties of sign pattern matrices, Linear Algebra Appl. 430 (2009) 3080–3094.
[3] R. Diestel, Graph Theory, Springer, 2005. 2005 Electronic edition available at <http://www.math.uni-hamburg.de/home/diestel/books/graph.theory/GraphTheoryIII.pdf>.
[4] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.
[5] W. Haemers, Matrices and graphs, in: L. Hogben (Ed.), Handbook of Linear Algebra, Chapman and Hall/CRC Press, Boca Raton, 2007
[6] F.J. Hall, Z. Li, Sign pattern matrices, in: L. Hogben (Ed.), Handbook of Linear Algebra, Chapman and Hall/CRC Press, Boca Raton, 2007
[7] N. Hindman, Partition regularity of matrices, in: B. Landman, M. Nathanson, J. Nesetril, R. Nowakowski, C. Pomerance (Eds.), Combinatorial Number Theory, deGruyter, Berlin, 2007, pp. 265–298.
[8] N. Hindman, I. Leader, D. Strauss, Open problems in partition regularity, Combin. Probab. Comput. 12 (2003) 571–583.
[9] S.V. Lokam, Complexity Lower Bounds using Linear Algebra, Now Publishers Inc., Hanover, MA, 2009.
[10] R. Rado, Studien zur kombinatorik, Math. Z. 36 (1933) 424–480.
[11] J.L. Stuart, Digraphs and matrices, in: L. Hogben (Ed.), Handbook of Linear Algebra, Chapman and Hall/CRC Press, Boca Raton, 2007.